Iterative method for three-point boundary value problem for third order differential equations

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Abstract

A third order three-point boundary value problem on a half-line is considered. Based on the application of lower and upper solutions an algorithm for constructing two monotone convergent sequences of successive approximations is provided. Both sequence are convergent and in the case their limits coincide this limit is a solution of the considered problem. Some examples are given to illustrate the suggested algorithm.

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1 Introduction

The third order differential equations arise in a wide variety of different areas of applied mathematics and physics, such as electromagnetic waves or gravity driven flows, a three layers beam, and the deflection of a curved beam having a constant or a varying cross section (for more details, see [8]). One of the most interesting problems about third order equations is the third order boundary value problem on infinite interval (see, for instance, [3, 4, 9, 15, 16] and references therein).

Most of the problems for third order differential equations are not possible to be obtained explicitly and it requires some approximate method to be proved. Note iterative techniques combined with lower and upper solutions are applied to approximately solve various problems for ordinary differential equations (see the classical monograph [14]), for second order periodic boundary value problems ([6]), for differential equations with maxima ([2, 7]), for impulsive differential equations ([5, 12]), for impulsive integro-differential equations ([10]), for impulsive differential equations with supremum ([11]), and for differential equations of mixed type ([13]).

The present paper deals with the three-point boundary value problem for the third order nonlinear scalar differential equation on a half line. Similar problem is studied in [1] and the existence of solutions is investigated based on given lower and upper solutions. The main contributions in the paper could be summarized as follows:

- an algorithm for construction of two monotone sequences of successive approximations is given;
- the convergence of both sequences is proved;
- both sequences approach their limits increasingly and decreasingly, respectively.

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– in the case both limits coincide, it is a solution of the studied problem.

Some examples are given to illustrate the application of the suggested iterative scheme.

2 Preliminaries

Consider the following third order nonlinear differential equation:

$$\begin{aligned} &x'''(t) + q(t)f(t, x(t), x'(t), x''(t) = 0, \quad t \in (0, +\infty), \\ &x'(0) = A, \quad x(\eta) = B, \quad x''(+\infty) = C, \end{aligned} \tag{2.1}$$

where $\eta > 0$ is a fixed number, $q \in C(\mathbb{R}_+, \mathbb{R}_+)$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R})$, $A,\ B \in \mathbb{R},\ C \geq 0$ are given constants.

Let

$$X = \left\{ x \in C^2(\mathbb{R}_+, \mathbb{R}) : \lim_{t\to\infty} \frac{x(t)}{1+t^2} < \infty, \lim_{t\to\infty} \frac{x'(t)}{1+t^2} < \infty, \lim_{t\to\infty} \frac{x''(t)}{1+t^2} < \infty \right\}.$$

We will define lower/upper solutions of the problem (2.1).

Definition 2.1. [1] A function $x \in X \cap C^3(\mathbb{R}_+, \mathbb{R})$ is called a lower (upper) solution of (2.1), if it satisfies

$$\begin{aligned} &x'''(t) + q(t)f(t, x(t), x'(t), x''(t) \geq (\leq) 0, \quad t \in (0, +\infty), \\ &x'(0) \leq (\geq) A, \quad x(\eta) = B, \quad x''(+\infty) \leq (\geq) C. \end{aligned} \tag{2.2}$$

Remark 2.2. [1] If the functions $\alpha,\ \beta \in X \cap C^3(\mathbb{R}_+, \mathbb{R})$ satisfy (2.2) and $\alpha'(t) \leq \beta'(t),\ t > 0$, then $\alpha(t) \geq \beta(t),\ t \in (0, \eta]$ and $\alpha(t) \leq \beta(t),\ t \in [\eta, \infty)$.

In our further investigations we will use the integral presentation of the solution of the following linear differential equation

$$\begin{aligned} &x'''(t) + v(t) = 0, \quad t \in (0, +\infty), \\ &x'(0) = A, \quad x(\eta) = B \quad x''(+\infty) = C. \end{aligned} \tag{2.3}$$

Lemma 2.3. [1]. Let $v \in C(\mathbb{R}_+, \mathbb{R})$ and $\int_0^\infty v(s)ds < \infty$. Then the function $x \in X \cap C^3(\mathbb{R}_+, \mathbb{R})$ is a solution of (2.3) iff

$$x(t) = B + A(t - \eta) + 0.5C(t^2 - \eta^2) + \int_0^\infty G(t, s)v(s)ds, \tag{2.4}$$

where

$$G(t, s) = \begin{cases} s(t - \eta) & s \leq \min\{\eta, t\}, \\ 0.5t^2 + 0.5s^2 - s\eta, & t \leq s \leq \eta, \\ st - 0.5s^2 - 0.5\eta^2, & \eta \leq s \leq t, \\ 0.5t^2 - 0.5\eta^2 & \max\{t, \eta\} \leq s, \end{cases} \tag{2.5}$$

or for $t \in [0, \eta]$

$$G(t,s) = \begin{cases} s(t-\eta) & s \le t, \\ 0.5t^2 + 0.5s^2 - s\eta, & t \le s \le \eta, \\ 0.5t^2 - 0.5\eta^2 & s \ge \eta, \end{cases} \tag{2.6}$$

and for $t \ge \eta$

$$G(t,s) = \begin{cases} s(t-\eta) & s \le \eta, \\ st - 0.5s^2 - 0.5\eta^2, & \eta \le s \le t, \\ 0.5t^2 - 0.5\eta^2 & s \ge t \le s. \end{cases} \tag{2.7}$$

Lemma 2.4. [1] The following inequalities

$$G(t,s) \begin{cases} \le 0, & t \in (0, \eta], \ s \ge 0, \\ \ge 0, & t \ge \eta, \ s \ge 0, \end{cases} \tag{2.8}$$

hold.

From Lemma 2.3 we obtain the following result:

Corollary 2.5. Let the function $x \in C^3(\mathbb{R}_+, \mathbb{R})$ satisfies the inequalities

$$\begin{aligned} &x'''(t) \le 0, \quad t \in (0, +\infty), \\ &x'(0) \le 0, \quad x(\eta) = 0 \quad x''(+\infty) \le 0. \end{aligned} \tag{2.9}$$

Then the inequality $x(t) \le 0$ holds for $t \ge 0$.

Definition 2.6. A function $x \in C^3(\bar{\mathbb{R}}_+, \mathbb{R}) \cap C^3(\mathbb{R}_+, \mathbb{R})$ is a mild solution of (2.1) if it satisfies the integral equality

$$x(t) = B + A(t-\eta) + 0.5C(t^2 - \eta^2) + \int_0^\infty G(t,s)q(s)f(s, x(s), x'(s), x''(s))ds, \tag{2.10}$$

where $G(t,s)$ is defined by (2.5).

Lemma 2.7. The function $x \in X \cap C^3(\mathbb{R}_+, \mathbb{R})$ is a mild solution of (2.1) iff it is a solution of (2.1).

Proof. Let $x \in X \cap C^3(\mathbb{R}_+, \mathbb{R})$ be a mild solution of (2.1). Differentiate (2.4) and obtain

$$\begin{aligned} x'(t) &= A + Ct + \int_0^\infty \frac{d}{dt} G(t,s)q(s)f(s, x(s), x'(s), x''(s))ds \\ &= A + Ct + \int_0^t sq(s)f(s, u(s), u'(s), u''(s))ds + t\int_t^\infty q(s)f(s, u(s), u'(s), u''(s))ds \\ x''(t) &= C + \int_t^\infty q(s)f(s, u(s), u'(s), u''(s))ds \\ x'''(t) &= -q(t)f(t, u(t), u'(t), u''(t))ds. \end{aligned} \tag{2.11}$$

Equation (2.11) proves the claim of Lemma 2.7.

Q.E.D.

3 Main results

We give an algorithm for constructing two sequences of successive approximations.

Theorem 3.1. Let the following conditions be fulfilled:

1. The functions $\alpha,\ \beta \in X \cap C^3(\mathbb{R}_+\mathbb{R})$ are a lower and upper solutions of (2.1) satisfying $\alpha'(t) \le \beta'(t),\ t \ge 0$.

2. The function $f \in C(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R})$ and

   (i) the inequality $f(t, \alpha(t), z, w) \le f(t, \beta(t), z, w)$ holds for $t \in [0, \infty),\ z \in [\alpha'(t), \beta'(t)]$, $w \in [\alpha''(t), \beta''(t)]$;

   (ii) there exist a constant $M_j > 0$, $j = 1,\ 2,\ 3$ such that for any $t \ge 0$ and $\max\{\alpha(t), \beta(t)\} \le x_1 \le x_2 \le \max\{\alpha(t), \beta(t)\}$, $y_2 \ge y_1,\ \ z_2 \ge z_1$ the inequality

   $$f(t, x_2, y_2, z_2) - f(t, x_1, y_1, z_1) \ge M_1(x_2 - x_1) + M_2(y_2 - y_1) + M_3(z_2 - z_1) \qquad (3.1)$$

   holds.

3. The function $q \in C(\mathbb{R}_+, \mathbb{R}_+)$.

Then there exist two sequences of functions $\{\alpha_n(t)\}_0^\infty$ and $\{\beta_n(t)\}_0^\infty$ such that:

a. The sequence $\{\alpha_n(t)\}_0^\infty$ is decreasing on $[0, \eta]$, i.e.

$$\alpha_n(t) \le \alpha_{n-1}(t) \quad \text{for } t \in [0, \eta], \quad n = 1,\ 2,\ \ldots$$

and it is increasing on $[\eta, \infty)$, i.e.

$$\alpha_n(t) \ge \alpha_{n-1}(t) \quad \text{for } t \ge \eta, \quad n = 1,\ 2,\ \ldots;$$

b. The sequence $\{\beta_n(t)\}_0^\infty$ is increasing on $[0, \eta]$, i.e.

$$\beta_n(t) \ge \beta_{n-1}(t) \quad \text{for } t \in [0, \eta], \quad n = 1,\ 2,\ \ldots$$

and it is decreasing on $[\eta, \infty)$, i.e.

$$\beta_n(t) \le \beta_{n-1}(t) \quad \text{for } t \ge \eta, \quad n = 1,\ 2,\ \ldots;$$

c. Both sequences converge on $[0, \infty)$, i.e. $\Phi(t) = \lim_{n\to\infty} \alpha_n(t)$, $\Psi(t) = \lim_{n\to\infty} \beta_n(t),\ t \in [0, \infty)$;

d. If the limit's functions $\Phi(t) \equiv \Psi(t)$ coincide, then they are a solution of the third order nonlinear differential equation (2.1).

Remark 3.2. If condition 1 is satisfied, then

$$\alpha(t) = \begin{cases} \ge \beta(t) & t \in (0, \eta], \\ \le \beta(t) & t \ge \eta. \end{cases} \qquad (3.2)$$

Proof. Define the operator $\mathcal{F}(u,v)$ for any $u,\ v \in X \cap C^3(\mathbb{R}_+,\mathbb{R})$

$$\mathcal{F}(u,v)(t) = \begin{cases} B + A(t-\eta) + 0.5C(t^2-\eta^2) \\ \quad + \displaystyle\int_0^\infty G(t,s)q(s)f(s,u(s),v'(s),v''(s))ds \quad \text{for} \quad t \in (0,\eta], \\ B + A(t-\eta) + 0.5C(t^2-\eta^2) \\ \quad + \displaystyle\int_0^\infty G(t,s)q(s)f(s,v(s),v'(s),v''(s))ds \quad \text{for} \quad t \geq \eta. \end{cases} \tag{3.3}$$

Define the sequences $\{\alpha_n(t)\}_{n=1}^{\infty},\ \ \{\beta_n(t)\}_{n=1}^{\infty}$ by the equalities $\alpha_0(t) \equiv \alpha(t),\ \beta_0(t) \equiv \beta(t)$ and

$$\alpha_n(t) = \mathcal{F}(\beta_{n-1},\alpha_{n-1})(t), \quad \beta_n(t) = \mathcal{F}(\alpha_{n-1},\beta_{n-1})(t), \quad n = 1,2,\dots. \tag{3.4}$$

Since the lower solution $\alpha_0(t)$ satisfies the inequality (2.2), there exists a function $\xi : [0,\infty) \to [0,\infty)$ and constants $\nu,\ \mu \geq 0$ such that

$$\begin{aligned} &\alpha_0'''(t) + q(t)f(t,\alpha_0(t),\alpha_0'(t),\alpha_0''(t) - \xi(t) = 0, \qquad t \in (0,+\infty), \\ &\alpha_0'(0) = A - \nu, \quad \alpha_0(\eta) = B, \quad \alpha_0''(+\infty) = C - \mu. \end{aligned} \tag{3.5}$$

Applying Lemma 2.7, the definition of $\alpha_1(t)$, the inequality (2.8), condition 2(i) and Lemma 2.4 we get for $t \in [0,\eta]$

$$\begin{aligned} \alpha_0(t) &= B + A(t-\eta) + 0.5C(t^2-\eta^2) - nu(t-\eta) - 0.5\mu(t^2-\eta^2) - \int_0^\infty G(t,s)\xi(s)ds \\ &\quad + \int_0^\infty G(t,s)q(s)f(s,\alpha_0(s),\alpha_0'(s),\alpha_0''(s))ds \\ &\geq B + A(t-\eta) + 0.5C(t^2-\eta^2) + \int_0^\infty G(t,s)q(s)f(s,\beta_0(s),\alpha_0'(s),\alpha_0''(s))ds = \alpha_1(t) \end{aligned}$$

and for $t \geq \eta$

$$\begin{aligned} \alpha_0(t) &= B + A(t-\eta) + 0.5C(t^2-\eta^2) - \nu(t-\eta) - 0.5\mu(t^2-\eta^2) - \int_0^\infty G(t,s)\xi(s)ds \\ &\quad + \int_0^\infty G(t,s)q(s)f(s,\alpha_0(s),\alpha_0'(s),\alpha_0''(s))ds \leq \alpha_1(t), \end{aligned}$$

or

$$\alpha_0(t) = \begin{cases} \geq \alpha_1(t) & t \in (0,\eta], \\ \leq \alpha_1(t) & t \geq \eta, \end{cases} \tag{3.6}$$

Applying

$$\frac{d}{dt}G(t,s) = \begin{cases} s & 0 \leq s \leq t, \\ t, & s \geq \eta, \end{cases} \qquad \frac{d^2}{dt^2}G(t,s) = \begin{cases} 0 & 0 \leq s \leq t, \\ 1, & s \geq \eta, \end{cases} \tag{3.7}$$

we get

$$\alpha_0'(t) - \alpha_1'(t) = -\nu - \mu t - \int_0^t s\xi(s)ds - t\int_t^\infty \xi(s)ds \leq 0,\ t \geq 0 \tag{3.8}$$

and

$$\alpha_0''(t) - \alpha_1''(t) = -mu - \int_t^{\infty} \xi(s)ds \le 0,\ t \ge 0. \tag{3.9}$$

Similarly, we prove that

$$\beta_0(t) = \begin{cases} \le \beta_1(t) & t \in (0, \eta], \\ \ge \beta_1(t) & t \ge \eta, \end{cases} \tag{3.10}$$

and $\beta_1'(t) \le \beta_0'(t)$, $\beta_1''(t) \le \beta_0''(t)$ on $[0, \infty)$.

Now, from inequalities (3.1) and $\beta_1 \ge \beta_0$, $\alpha_1' \ge \alpha_0'$, $\alpha_1'' \ge \alpha_0''$ on $[0, \eta]$ we obtain

$$\begin{aligned} &\alpha_2(t) - \alpha_1(t) = \int_0^{\infty} G(t,s)q(s)[f(s, \beta_1(s), \alpha_1'(s), \alpha_1''(s)) - f(s, \beta_0(s), \alpha_0'(s), \alpha_0''(s))]ds \\ &\le \int_0^{\infty} G(t,s)q(s)[M_1(\beta_1(s) - \beta_0(s)) + M_2(\alpha_1'(s) - \alpha_0'(s)) + M_3(\alpha_1''(s) - \alpha_0''(s))]ds \\ &\le 0, \quad t \in [0, \eta], \end{aligned} \tag{3.11}$$

and from inequalities (3.1) and $\alpha_1 \ge \alpha_0$, $\alpha_1' \ge \alpha_0'$, $\alpha_1'' \ge \alpha_0''$ on $[\eta, \infty)$ we get

$$\begin{aligned} &\alpha_2(t) - \alpha_1(t) = \int_0^{\infty} G(t,s)q(s)[f(s, \alpha_1(s), \alpha_1'(s), \alpha_1''(s)) - f(s, \alpha_0(s), \alpha_0'(s), \alpha_0''(s))]ds \\ &\ge \int_0^{\infty} G(t,s)q(s)[M_1(\alpha_1(s) - \alpha_0(s)) + M_2(\alpha_1'(s) - \alpha_0'(s)) + M_3(\alpha_1''(s) - \alpha_0''(s))]ds \\ &\ge 0, \quad t \ge \eta. \end{aligned} \tag{3.12}$$

Also, for $t \in [0, \eta]$

$$\alpha_2'(t) - \alpha_1'(t) = \int_0^{\infty} \frac{d}{dt} G(t,s)q(s)[f(s, \beta_1(s), \alpha_1'(s), \alpha_1''(s)) - f(s, \beta_0(s), \alpha_0'(s), \alpha_0''(s))]ds \ge 0,$$

and for $t \in [\eta, \infty)$

$$\alpha_2'(t) - \alpha_1'(t) = \int_0^{\infty} \frac{d}{dt} G(t,s)q(s)[f(s, \alpha_1(s), \alpha_1'(s), \alpha_1''(s)) - f(s, \alpha_0(s), \alpha_0'(s), \alpha_0''(s))]ds \ge 0.$$

Therefore, we proved that

$$\alpha_2(t) = \begin{cases} \ge \alpha_1(t) & t \in (0, \eta], \\ \le \alpha_1(t) & t \ge \eta, \end{cases} \tag{3.13}$$

and $\alpha_2'(t) \ge \alpha_1'(t)$ and $\alpha_2''(t) \ge \alpha_1''(t)$ for $t \ge 0$.

Similarly, we prove

$$\beta_2(t) \begin{cases} \le \beta_1(t) & t \in (0, \eta], \\ \ge \beta_1(t) & t \ge \eta, \end{cases} \tag{3.14}$$

$\beta_0'(t) \le \beta_1'(t),\ \beta_0''(t) \le \beta_1''(t),\ t \ge 0.$

Using induction we prove for $n = 1, 2, \dots$ the inequalities

$$\alpha_n(t) \begin{cases} \ge \alpha_{n-1}(t) & t \in (0, \eta], \\ \le \alpha_{n-1}(t) & t \ge \eta, \end{cases} \qquad \beta_n(t) \begin{cases} \le \beta_{n-1}(t) & t \in (0, \eta], \\ \ge \beta_{n-1}(t) & t \ge \eta. \end{cases} \tag{3.15}$$

For $t \in [0, \eta]$ we obtain

$$\begin{aligned} \alpha_1(t) - \beta_1(t) &= \int_0^\infty G(t,s)q(s)[f(s, \beta_0(s), \alpha_0'(s), \alpha_0''(s)) - f(s, \alpha_0(s), \beta_0'(s), \beta_0''(s))]ds \\ \le \int_0^\infty & G(t,s)q(s)[M_1(\beta_0(s) - \alpha_0(s)) + M_2(\alpha_1'(s) - \alpha_0'(s)) + M_3(\alpha_1''(s) - \alpha_0''(s))]ds \\ \le 0, \quad & t \in [0, \eta]. \end{aligned} \tag{3.16}$$

Therefore, the clams a/ and b/ are proved.

Both sequences of functions $\{\alpha_{(n)}(t)\}_0^\infty$ and $\{\beta_{(n)}(t)\}_0^\infty$ being monotonic and bounded are uniformly convergent on $[0, \infty)$. Let $\Phi(t) = \lim_{n\to\infty} \alpha_n(t)$, $\Psi(t) = \lim_{n\to\infty} \beta_n(t)$, $t \in [0, \infty)$. Take the limit in the equalities (3.4) and obtain the integral equations

$$\Phi(t) = B + A(t - \eta) + 0.5C(t^2 - \eta^2) + \int_0^\infty G(t,s)q(s)f(s, \Psi(s), \Phi'(s), \Phi''(s))ds,\ t \in [0, \eta]$$

and

$$\Phi(t) = B + A(t - \eta) + 0.5C(t^2 - \eta^2) + \int_0^\infty G(t,s)q(s)f(s, \Phi(s), \Phi'(s), \Phi''(s))ds,\ t \ge \eta.$$

Similarly, we get

$$\Psi(t) = B + A(t - \eta) + 0.5C(t^2 - \eta^2) + \int_0^\infty G(t,s)q(s)f(s, \Psi(s), \Psi'(s), \Psi''(s))ds,\ t \in [0, \eta]$$

and

$$\Psi(t) = B + A(t - \eta) + 0.5C(t^2 - \eta^2) + \int_0^\infty G(t,s)q(s)f(s, \Phi(s), \Psi'(s), \Psi''(s))ds,\ t \ge \eta.$$

If both limits coincide, i.e. $\Phi(t) \equiv \Psi(t),\ t \ge 0$, then both function are a solution of the third order nonlinear differential equation (2.1).

Q.E.D.

4 Applications

Initially, we will start with a simple example:

Example 4.1. Consider the following nonlinear equation

$$\begin{aligned} & x'''(t) + \frac{(x - 2 + t)^2}{e^{t+10} + 2} = 0 \text{ for } t \in [0, \infty), \\ & x'(0) = -1, \quad x(\eta) = 2 - \eta, \quad x''(\infty) = 0. \end{aligned} \tag{4.1}$$

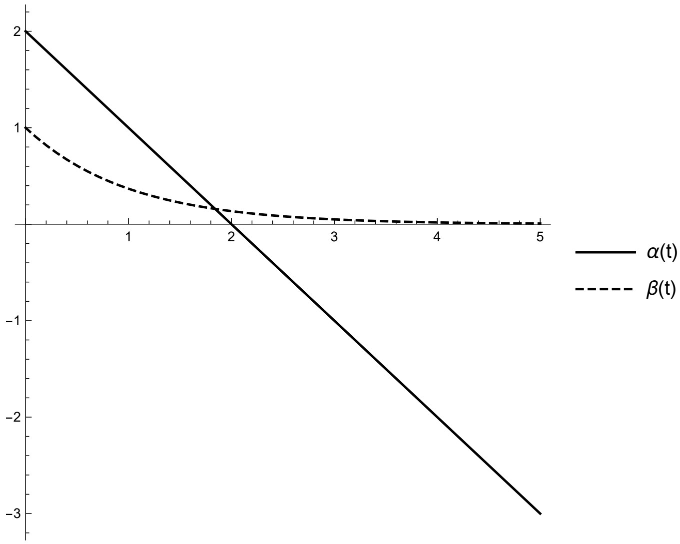

Figure 1. Graphs of $\alpha(t)$ and $\beta(t)$ for $t \in [0, 5]$.

Let $\eta \in (1.5, 2]$ be the solution of $2 - \eta = e^{-\eta}$, or $\eta \approx 1.84141$.

Choose $\alpha(t) = 2 - t$ and $\beta(t) = e^{-t}$. The $\alpha(t) = 2 - t \geq \beta(t) = e^{-t}$ on $[0, \eta]$ and $\alpha(t) = 2 - t \leq \beta(t) = e^{-t}$ on $t \geq \eta$ (see Figure 1).

Also,

$$\alpha'''(t) + \frac{(\alpha(t) - 2 + t)^2}{e^{t+10} + 2} = 0,$$

$\alpha'(0) = -1 \leq -1 = A$, $\alpha(\eta) = 2 - \eta = B$, $\alpha''(\infty) = 0 = C$ and

$$\beta'''(t) + \frac{(\beta(t) - 2 + t)^2}{e^{t+10} + 2} = -e^{-t} + \frac{(e^{-t} - 2 + t)^2}{e^{t+10} + 2} \leq 0$$

$\beta'(0) = -1 \geq -1 = A$, $\beta(\eta) = 2 - \eta = B$, $\beta''(\infty) = 0 = C$

Also, $f(t, \alpha(t)) = 0 \leq f(t, \beta(t)) = \dfrac{(e^{-t} - 2 + t)^2}{e^{t+10} + 2}$, $t \geq 0$.

Therefore, α is a lower solution, β is an upper solution.

Construct the following functions $\alpha_n(t)$, $n = 1, 2, 3, 4, \ldots$ by:

$$\alpha_n(t) = \begin{cases} 2 - t + (t-\eta)\displaystyle\int_0^t s\frac{(\beta_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_t^\eta (0.5t^2+0.5s^2-s\eta)\frac{(\beta_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_\eta^\infty (0.5t^2-0.5s^2)\frac{(\beta_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds, & t\in(0,\eta], \\ 2 - t + (t-\eta)\displaystyle\int_0^\eta s\frac{(\alpha_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_\eta^t (st-0.5s^2-0.5\eta^2)\frac{(\alpha_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_t^\infty (0.5t^2-0.5s^2)\frac{(\alpha_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds, & t\geq\eta, \end{cases}$$

and construct the following functions $\beta_n(t),\ n = 1, 2, 3, 4, \ldots$ by:

$$\beta_n(t) = \begin{cases} 2 - t + (t-\eta)\displaystyle\int_0^t s\frac{(\alpha_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_t^\eta (0.5t^2+0.5s^2-s\eta)\frac{(\alpha_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_\eta^\infty (0.5t^2-0.5s^2)\frac{(\alpha_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds, & t\in(0,\eta], \\ 2 - t + (t-\eta)\displaystyle\int_0^\eta s\frac{(\beta_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_\eta^t (st-0.5s^2-0.5\eta^2)\frac{(\beta_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_t^\infty (0.5t^2-0.5s^2)\frac{(\beta_{n-1}(s)-2+s)^2}{e^{s+10}+2}ds, & t\geq\eta. \end{cases}$$

For example, the first two successive approximations are given by:

$$\alpha_1(t) = \begin{cases} \alpha_0(t) + (t-\eta)\displaystyle\int_0^t s\frac{(e^{-s}-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_t^\eta (0.5t^2+0.5s^2-s\eta)\frac{(e^{-s}-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_\eta^\infty (0.5t^2-0.5s^2)\frac{(e^{-s}-2+s)^2}{e^{s+10}+2}ds \leq \alpha_0(t), & t\in(0,\eta], \\ 2-t = \alpha_0(t), & t\geq\eta, \end{cases}$$

and

$$\beta_1(t) = \begin{cases} 2-t = \alpha_0(t), & t\in(0,\eta], \\ 2 - t + (t-\eta)\displaystyle\int_0^\eta s\frac{(e^{-s}-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_\eta^t (st-0.5s^2-0.5\eta^2)\frac{(e^{-s}-2+s)^2}{e^{s+10}+2}ds \\ \quad + \displaystyle\int_t^\infty (0.5t^2-0.5s^2)\frac{(e^{-s}-2+s)^2}{e^{s+10}+2}ds, & t\geq\eta, \end{cases}$$

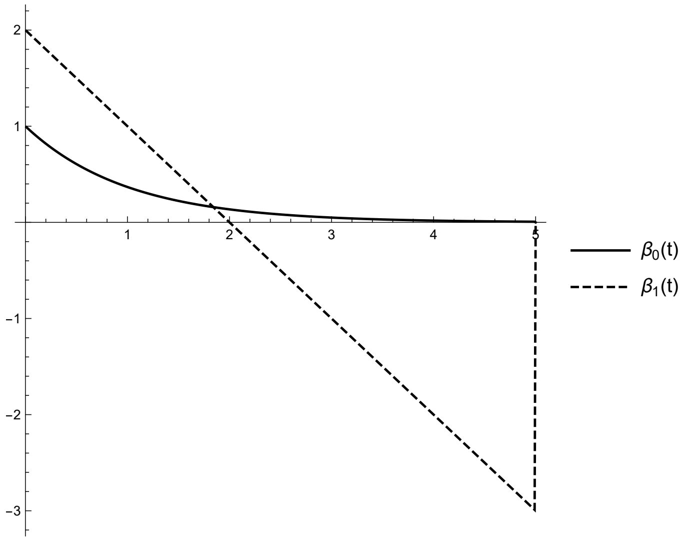

Figure 2. Graphs of $\beta_0(t)$ and $\beta_1(t)$ for $t \in [0,5]$.

$$\alpha_2(t) = \begin{cases} 2 - t + \int_\eta^\infty (0.5t^2 - 0.5s^2)\frac{(\beta_1(s) - 2 + s)^2}{e^{s+10} + 2}ds \le \alpha_0(t), & t \in (0,\eta], \\ 2 - t + (t-\eta)\int_0^\eta s\frac{(\alpha_1(s) - 2 + s)^2}{e^{s+10} + 2}ds & \\ \quad + \int_\eta^t (st - 0.5s^2 - 0.5\eta^2)\frac{(\alpha_1(s) - 2 + s)^2}{e^{s+10} + 2}ds, & t \ge \eta, \end{cases}$$

and

$$\beta_2(t) = \begin{cases} 2 - t + (t-\eta)\int_0^t s\frac{(\alpha_1(s) - 2 + s)^2}{e^{s+10} + 2}ds & \\ \quad + \int_t^\eta (0.5t^2 + 0.5s^2 - s\eta)\frac{(\alpha_1(s) - 2 + s)^2}{e^{s+10} + 2}ds, & t \in (0,\eta], \\ 2 - t + (t-\eta)\int_0^\eta s\frac{(\alpha_1(s) - 2 + s)^2}{e^{s+10} + 2}ds & \\ \quad + \int_\eta^t (st - 0.5s^2 - 0.5\eta^2)\frac{(\alpha_1(s) - 2 + s)^2}{e^{s+10} + 2}ds, & t \ge \eta. \end{cases}$$

Note the approximations in this example approaches to each other very fast and the graphs of $\beta_0(t)$ and $\beta_1(t)$ could be seen on Figure 2.

Example 4.2. In paper [1] the following nonlinear equation is studied

$$\begin{aligned} &x'''(t) + \frac{1}{1-t^9}(x(t)-1)^2(2t - x'(t))^2(1 + \sin(x''(t))^2) = 0 \text{ for } t \in [0,\infty), \\ &x'(0) = 0, \quad x(1) = 1, \quad x''(\infty) = 0. \end{aligned} \tag{4.2}$$

Choose $\alpha_0(t) = 2 - t^2$ and $\beta_0(t) = t^2$ as a lower and an upper solutions of (4.2). In [1] it is proved the existence of solutions of (4.2). Now, applying the suggested algorithm we construct the successive approximations by

$$\alpha_n(t) = \begin{cases} 2 - t + (t-\eta)\displaystyle\int_0^t s\frac{1}{1-s^9}(\beta_{n-1}(s)-1)^2(2s-\beta'_{n-1}(s))^2(1+\sin(\beta''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_t^\eta (0.5t^2+0.5s^2-s\eta)\frac{1}{1-s^9}(\beta_{n-1}(s)-1)^2(2s-\beta'_{n-1}(s))^2(1+\sin(\beta''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_\eta^\infty (0.5t^2-0.5s^2)\frac{1}{1-s^9}(\beta_{n-1}(s)-1)^2(2s-\beta'_{n-1}(s))^2(1+\sin(\beta''_{n-1}(s))^2)ds \\ \qquad\qquad t \in (0,\eta],\ n = 1,2,\dots, \\ 1 + (t-\eta)\displaystyle\int_0^\eta s\frac{1}{1-s^9}(\alpha_{n-1}(s)-1)^2(2s-\alpha'_{n-1}(s))^2(1+\sin(\alpha''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_\eta^t (st-0.5s^2-0.5\eta^2)\frac{1}{1-s^9}(\alpha_{n-1}(s)-1)^2(2s-\alpha'_{n-1}(s))^2(1+\sin(\alpha''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_t^\infty (0.5t^2-0.5s^2)\frac{1}{1-s^9}(\alpha_{n-1}(s)-1)^2(2s-\alpha'_{n-1}(s))^2(1+\sin(\alpha''_{n-1}(s))^2)ds \\ \qquad\qquad t \geq \eta,\ n = 1,2,\dots, \end{cases}$$

and construct the following functions $\beta_n(t),\ n = 1,2,3,4,\dots$ by:

$$\beta_n(t) = \begin{cases} 1 + (t-\eta)\displaystyle\int_0^t s\frac{1}{1-s^9}(\alpha_{n-1}(s)-1)^2(2s-\alpha'_{n-1}(s))^2(1+\sin(\alpha''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_t^\eta (0.5t^2+0.5s^2-s\eta)\frac{1}{1-s^9}(\alpha_{n-1}(s)-1)^2(2s-\alpha'_{n-1}(s))^2(1+\sin(\alpha''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_\eta^\infty (0.5t^2-0.5s^2)\frac{1}{1-s^9}(\alpha_{n-1}(s)-1)^2(2s-\alpha'_{n-1}(s))^2(1+\sin(\alpha''_{n-1}(s))^2)ds \\ \qquad\qquad t \in (0,\eta],\ n = 1,2,\dots, \\ 1 + (t-\eta)\displaystyle\int_0^\eta s\frac{1}{1-s^9}(\beta_{n-1}(s)-1)^2(2s-\beta'_{n-1}(s))^2(1+\sin(\beta''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_\eta^t (st-0.5s^2-0.5\eta^2)\frac{1}{1-s^9}(\beta_{n-1}(s)-1)^2(2s-\beta'_{n-1}(s))^2(1+\sin(\beta''_{n-1}(s))^2)ds \\ \quad + \displaystyle\int_t^\infty (0.5t^2-0.5s^2)\frac{1}{1-s^9}(\beta_{n-1}(s)-1)^2(2s-\beta'_{n-1}(s))^2(1+\sin(\beta''_{n-1}(s))^2) \\ \qquad\qquad t \geq \eta,\ n = 1,2,\dots. \end{cases}$$

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References

[1] R. Agarwal, E. Cetin, *Unbounded solutions of third order three-point boundary value problems on a half-line*, Adv. Nonlinear Anal., May 2015, DOI: 10.1515/anona-2015-0043.

[2] R. Agarwal, S. Hristova, *Quasilinearization for initial value problems involving differential equations with "maxima"*, Math. Comput. Modell., (2012), **55**, 9–10, 2096–2105.

[3] R. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordercht, (2001).

[4] C. Bai, C. Li, *Unbounded upper and lower solution method for third-order boundary-value problems on the half-line*, Electron. J. Diff. Eq., (2009), **119**, 1–12.

[5] D. Bainov, S. Hristova, *The method of quasilinearization for the periodic boundary value problem for systems of impulsive differential equations*, Appl. Math. Comput., (2001), **117**, 1, 73–85.

[6] A. Cabada, J. Nieto, *A generalization of the monotone iterative technique for nonlinear second order periodic boundary value problems*, J. Math. Anal. Appl., (1990), **151**, 1, 181–189.

[7] A. Golev , S. Hristova, A. Rahnev, *An algorithm for approximate solving of differential equations with "maxima"*, Comput. Math. Appl., (2010), **60**, 10, 2771–2778.

[8] M. Gregus, *Third order linear differential equations*, Math. Appl. (East Eur. Ser.), 22, Reidel Publishing, Dordrecht, (1987).

[9] J. Guo, J. Tsai, *The structure of solutions for a third order differential equation in boundary layer theory*, Japan J. Industrial Appl. Math., (2005), **22**, 311–351.

[10] Z. He, X. He, *Monotone Iterative Technique for Impulsive Integro-Differential Equations with Periodic Boundary Conditions*, Comput. Math. Appl., (2004), **48**, 73–84.

[11] S. Hristova, D. Bainov, *Monotone-Iterative Techniques of V. Lakshmikantham for a Boundary Value Problem for Systems of Impulsive Differential Equations with supremum*, J. Math. Anal. Appl., (1993), **172**, 339–352.

[12] P. Eloe, S. Hristova, *Method of the quasilinearization for nonlinear impulsive differential equations with linear boundary conditions*, Electron. J. Qual. Theory Differ. Equ., (2002), **10**, 1–14.

[13] T. Jankowski, *Boundary value problems for first order differential equations of mixed type*, Nonl. Anal.: Theory, Methods, Appl., (2006), **64**, 9, 1984-1997.

[14] G. Ladde, V. Lakshmikantham, A. Vatsala, *Monotone iterative techniques for nonlinear differential equations*, Pitman Advanced Publishing Program, (1985).

[15] H. Lian, J. Zhao, *Existence of unbounded solutions for third-order boundary value problem on infinite intervals*, Discrete Dyn. Nat. Soc., (2012), **2012**, Article ID 357697.

[16] P. Palamides, R. Agarwal, *An existence theorem for a singular third-order boundary value problem on* $[0, +\infty)$, Appl. Math. Lett., (2008), **21**, 1254–1259.